CONSIDERATIONS ON THE FEEDBACK THEORY

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A generalization of the elementary feebdack theory, which makes it applicable to any type of network, is proposed. Such an extention is based on a proper circuit decomposition, that is on cutting the network and on compensating the cut with an impedance and a generator such that the network currents and voltages are not changed. Then the analysis is made directly on this equivalent open-loop circuit by means of the normal theories of the four-terminal networks and of the transfer functions. Most results of the Bode general feebdack theory are so reached in a new, exact and intuitive way.

1. – Introduction.

As well known feedback theory is a fundamental tool of analysis, synthesis and design of the systems and, in particular, of the linear electronic networks. The feedback may be purposefully introduced in a system in order to reach some important properties and purposes such as the reduction of the system sensitivity to changes of its parameters, the simplification of the synthesis of a given transfer function, the control of the input and output impedances, the control of the stability, or instability, of the system and so on.

There are two ways to approach and devolop the feedback theory, one elementary and one general. The elementary theory expresses the quantities relative to a feedback network, such as the loop gain, the return difference, the sensitivity and the over-all system gain, by means of the transfer functions A and β of the forward and backward circuits, respectively, in the case that these are « separate » and A and β may be calculated « independently » of one another. When such « separation » does not exist and ambiguities arise in the A and β evaluation and also when the leakage trasmission from imput to output cannot be neglected, one must leave the intuitive elementary theory and use the general one.

The general feedback theory, all its most important theorems are due to H. W. Bode [1] who analyses the network as a whole, without distinction between forward and backward circuits, in terms of nodal equations and of the corresponding determinants. But in this way most of the intuitive concepts of the elementary theory are lost.

Such important intuitive features have been then newly introduced in the general theory and the Bode theorems are rederived in a much simpler way [2, 3, 4] by means of the signal-flow diagrams originated by S. J. Mason [4]. In particular S. Barabaschi and E. Gatti [3, 2], by using both a generalised nodal analysis and the signal-flow diagrams, have shown that the quantities of the Bode general feedback theory, i.e. the loop gain, the return difference and so on, relative for instance to a given active element, may be expressed by means of transfer functions, evaluated when the active element itself is « properly made passive », analogous to ones of the elementary theory.

In this paper, by developing such a point of view, it is shown that the elementary feedback theory may be exentended and made general, like the Bode theory, without losing its intuitive meaning. This result is achieved by means of a proper circuit decomposition, i.e. by cutting the network and by applying to the cut so obtained an impedance and a signal generator in such a way that the new network is equivalent to the original one. Then the analysis is easily made on this new open-loop circuit — which has all its elements, active and passive, « living » — by means of the normal theories of the four-terminal networks and of the transfer functions. In this way a general method and exact definitions, which allow one to use the elementary feedback theory in all cases, for any type of network, are given.

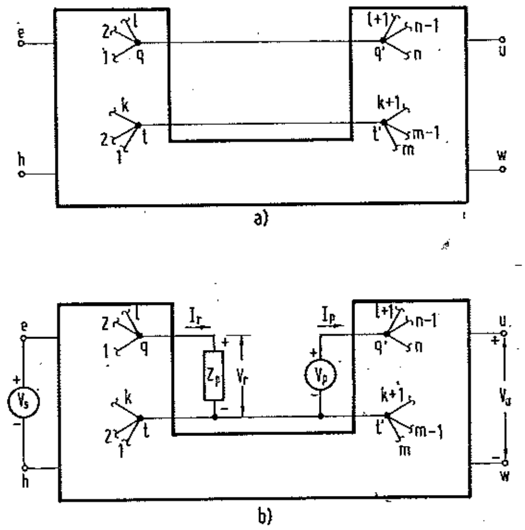

Fig. 1. — *a)* A given linear active network; *b)* its equivalent circuit after cutting the connection $q\,q'$.

2. – Feedback theory as a general network analysis tool.

a) *Network Decomposition.*

In a given linear network (fig. 1 *a*) let h, e, u, w, $q \equiv q'$ and $t \equiv t'$ be six arbitrary nodes. Let the first purpose of the network analysis be the determination of the transfer function between the ports $e\,h$ and $u\,w$.

For this one of the two connections $q\,q'$ and $t\,t'$, for instance $q\,q'$, is broken (the separation 1, ..., l and $l+1$, ..., n of the n branches which arrive in the node $q \equiv q'$ of fig. 1 *a* is arbitrary), an arbitrary impedance Z_p is put between q and t and abitrary independent

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ideal voltage sources V_p and V_s are put between q' and t' and between e and h, respectively, as indicated in fig. 1 *b*. (V_p and V_s are the only independent external signal sources of the network. Of course one may put V_p between q and t and Z_p between q' and t').

With reference to the « open-loop » network of fig. 1 *b* one defines the following transfer functions and impedances of its various four-terminal networks:

$$A = \left.\frac{V_u}{V_p}\right|_{V_s = 0}, \quad \beta = \left.\frac{V_r}{V_u}\right|_{V_s = 0}, \tag{1}$$

$$Z_t = \frac{1}{G_t} = \left.\frac{V_p}{I_p}\right|_{V_s = 0},$$

$$\gamma = \left.\frac{V_u}{V_s}\right|_{V_p = 0}, \quad \alpha = \left.\frac{V_r}{V_s}\right|_{V_p = 0}, \tag{2}$$

$$\varrho = \left.\frac{I_p}{V_s}\right|_{V_p = 0},$$

which are functions of Z_p.

The voltages V_r and V_u and the currents I_r and I_p due to the two voltage sources V_p and V_s acting simultaneously, in virtue of the system linearity and of the inpedendence of V_s and V_p between themselves, which make applicable the superimposition theorem, may be expressed by means of the transfer functions and of the impedances defined by (1) and (2) in the form:

$$V_r = Z_p I_r = \alpha V_s + A \beta V_p, \tag{3}$$

$$V_u = \gamma V_s + A V_p, \tag{4}$$

$$I_p = \varrho V_s + G_t V_p, \tag{5}$$

Now it is possible to prove that the network of fig. 1 *a* (with the generator V_s put between e and h) and the one of fig. 1*b* are quivalent when the arbitrary impedance Z_p and voltage source V_p are chosen in such a way as to make simultaneously:

$$V_r = V_p, \tag{6}$$

$$I_r = I_p, \tag{7}$$

that is, for (3) and (5), when V_p and $Z_p = 1/G_p$ have, respectively, the values given by the equations:

$$V_p = \frac{\alpha}{1 - A\beta} V_s, \tag{8}$$

$$G_p = G_t + \frac{\varrho}{\alpha}(1 - \beta A), \tag{9}$$

which hold if $1 - \beta A$ and α are not zero. In fact the nodal equations of the nodes of the two networks of fig. 1 which are distinct from $t \equiv t'$, q and q', for (6), are identical just as, for (7) and (6), the equation of the node $t \equiv t'$ is the same for the two cases. At the same time if, from the two nodal equations relative to the nodes q and q' of fig. 1 *b*, one eliminates $I_p = I_r$, for (6), one obtains an equation equal to the nodal one concerning the node $q \equiv q'$ of fig. 1 *a*. Therefore, the equation systems of the two networks being identical, for the uniqueness of the solution of a linear algebrical equation system the voltages are equal in the two networks.

In order to determine the impedance $Z_p = 1/G_p$ one must solve the eq. (9) in which all the quantities are functions of Z_p itself. Such a problem is remarkably simplified when $\varrho = 0$ that is when the four-terminal network $q' t'$, $e h$ is unidirectional. In this case Z_p becomes equal to the iterative impedance Z_t of the four-terminal network $q' t'$, $q t$; in its turn Z_t results independent from Z_p when also the network $q' t'$, $q t$ is unidirectional.

b) *Over-All System Gain.*

In virtue of the equivalance, proved in the preceding section, between the networks of fig. 1 one may deduce the properties of the original network of fig. 1 *a* from the analysis of the « open-loop » one of fig. 1 *b*.

In particular from (8) and (4) (which holds for any value of V_p and Z_p) the over-all system gain $A_f = V_u/V_s$ from imput to output becomes:

$$A_f = \frac{\alpha A}{1 - \beta A} + \gamma, \tag{10}$$

that is, as in the elementary feedback theory, the transfer gain A_f is expressed by means of the transfer functions A and β of the forward and backward circuits, respectively, and by means of the leakage-transmission term γ and of the transfer function α between the ports $e\,h$ and $q\,t$ (fig. 1 *b*).

In the present analysis the feedback is referred to a given pair of nodes $q \equiv q'$ and $t \equiv t'$ in the sense that the feedback exists with respect to them if the voltage $V_r = V_p$ may be expressed as a linear combination of the imput and output voltages V_s and V_u, respectively, that is if, simultaneously, $\alpha \neq 0$ and $\beta \neq 0$.

Instead in the theory of Bode [1] and of the other authors [2, 3, 5] the feedback is considered around an element and its value, for instance the transconductance g_m of an active element, is put in explicit evidence in the various quantities, as in the return ratio or loop gain βA.

Also in the present method one may do the same thing. In fact if the nodes $q \equiv q'$ and $t \equiv t'$ are selected in such a way that the voltage $V_r = V_p$ between them is the voltage on which the generator $g_m V_p$ of the active element depends and the branches $l + 1, \ldots, n$ (fig. 1 *a*) are all on the q side, it is $\varrho = 0$, $Z_p = Z_t = \infty$ and:

$$A = g_m A_1, \tag{11}$$

where A_1 is the value given by the first of (1) for $g_m = 1$ A/V.

When $\gamma \neq 0$ one can put $\alpha = \gamma \beta^*$ where $\beta^* = (V_r/V_u)|_{V_p = 0}$. If $\beta^* = \beta$, (10) becomes:

$$A_f = \frac{\gamma}{1 - \beta A}. \tag{12}$$

One may have $\beta^* = \beta$ for instance when the generator V_s, which can represent an external signal, the initial conditions in the complex frequency domain, a disturbing signal and so on, has the terminals e and h on the forward line (assumed distinct from the backward one) between the output port $u\ w$ and the reference one $q'\ t'$.

c) *Sensitivity and Return Difference.*

In the preceding section the return ratio or loop gain βA, the return difference $F_{qt} = 1 - \beta A$ and the over-all transmittance A_f of the network of fig. 1 *a* have been expressed, with respect to the control node $q \equiv q'$ and to the reference one $t \equiv t'$, directly by means of the transfer functions A, α, β and γ of the four-terminal networks of the open-loop circuit of fig. 1 *b* as in the elementary theory, without using the signal-flow diagrams and/or the nodal equation systems and the relative determinants.

Now, by following Mason's definition [4, 5] of sensitivity $S^{A_f}{}_W$ of the over-all trasmittance A_f with respect to a given parameter W, one has:

$$(13) \qquad S^{A_f}{}_W = \frac{W}{A_f} \, \frac{dA_f}{dW},$$

where W may be equal to A, β, α an γ or, for (11), to g_m.

From (13) and (10) one obtains the relationships between $S^{A_f}{}_W$, F_{qt} and the other parameters, deduced by Bode [1] (who defines as sensitivity $1/S^{A_f}{}_W$) and Mason

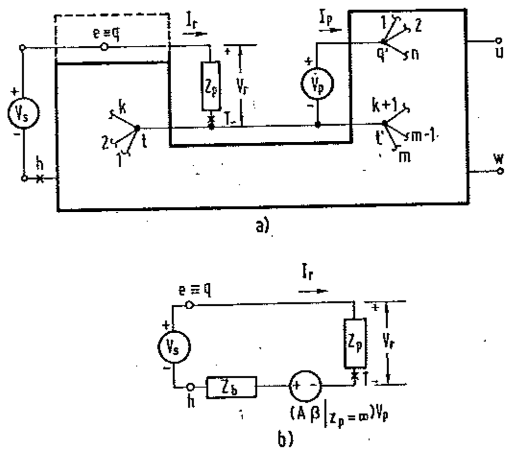

Fig. 2. — *a*) Coincidence of the nodes *e* and *q* in the evaluation of the impedance between the nodes *e* and *h*; *b*) its Thèvenin equivalent circuit.

[4, 5]. In particular when the leakege transmittance γ is zero $S^{A_f}{}_A = S^{A_f}{}_{gm} = 1/F_{qt}$, whereas, for $|\beta A| \gg 1$, $S^{A_f}{}_\beta = 1$ as in the elementary theory.

d) *Impedances.*

The equivalence between the circuits of fig. 1 allows one to calculate in a general and simple way the impedance seen looking into the network through the arbitrary pair of nodes e and h (e and h may be also the terminals of a branch broken, just in order to calculate the impedance seen looking into the circuit through them).

For this purpose in fig. 1 one chooses q coinciding with e as indicated in fig. 2 *a* where moreover all the branches distinct from that relative to the external generator V_s are brought in q'.

From fig. 2 *a* the impedance Z between e and h is given by $Z = V_s/I_r$ and, for (3), (6) and (8), it becomes:

$$(14) \qquad Z = Z_p \frac{(1 - \beta A)}{\alpha},$$

By applying the Thévenin theorem to the network of fig. 2 *a*, between h and T, one obtains the equivalent circuit of fig. 2 *b* from which and from the second definition (2) it results $\alpha = Z_p/(Z_p + Z_b)$ so that Z becomes:

$$(15) \qquad Z = (1 - \beta A)(Z_p + Z_b).$$

From fig. 2 *b*, in which $(A\ \beta\ |_{Z_p = \infty}) = (V_r / V_p)\ |_{V_s = 0,\, Z_p = \infty}$ is the loop gain for $Z_p = \infty$, and from the A and β definitions given by (1) it is also:

$$(16) \qquad Z = Z_b + Z_p\,(1 - A\ \beta\,|_{Z_p = \infty}).$$

In particular the eq. (14) or (15) or (16) may be used to calculate both the imput and output impedances of a given system.

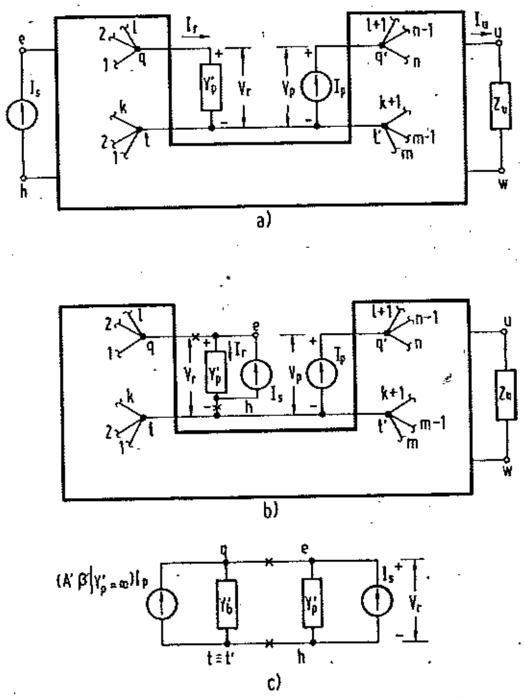

Fig. 3. — *a*) Dual case of the one of fig. 1 *b*; *b*) coincidences $e \equiv \equiv q$ and $h \equiv t \equiv t'$ in the calculation of the admittance between the nodes *e* end *h*; *c*) corresponding Norton equivalent circuit.

e) *Dual Case.*

The preceding analysis can be made by using an ideal current source I_p in place of the voltage one V_p and by modifying properly the definitions (1) and (2).

In particular if the interest is in determining the current gain $A'_f = I_u/I_s$, that is the ratio between the current I_u which passes through the impedance Z_u put between u and w and the current I_s due to the external generator, one may use the circuit configuration of fig. 3 *a* which is dual to the one of fig. 1 *b*. The analysis is completely dual to the preceding one, i.e. in the various equations each voltage and the impedance must be replaced by the corresponding current and admittance, respectively, and viceversa.

The admittance Y' between the nodes e and h may be obtained from the dual equations of (14), (15) and (16) by making the nodes q and $t \equiv t'$ coincide with e and h, respectively, as indicated in fig. 3 b and 3 c where Y'_b is the Norton equivalent admittance of the network external to Y'_p and I_s. In this case $A' \beta' |_{Y'_p = \infty}$ is the loop gain $(I_r/I_p) |_{I_s = 0,\, Y'_p = \infty}$ for $Z'_p = 1/Y'_p = 0$. In particular, by rewriting (16), it results:

$$Y' = Y'_b + Y'_p \left(1 - A' \beta' |_{Y'_p = \infty}\right) . \tag{17}$$

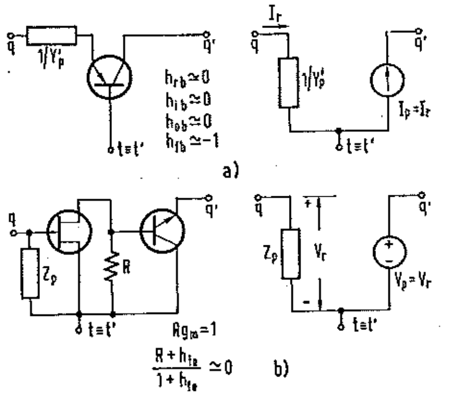

Fig. 4. — Three-terminal networks, and their a.c. equivalent circuits, which may be inserted among the nodes q, q' and $t \equiv t'$ (after the cutting of the connection $q\,q'$) of the circuit of fig. 1 a without changing its voltages.

It may be useful to note that, for the equivalence between the network of fig. 3 a and the uncut original one, in the latter one may introduce, among q, q' and $t \equiv t'$, the three-terminal network of fig. 4 a obtaining in this way a new equivalent network. In an analogous way one may introduce the circuit of fib. 4 b in the network of fig. 1 a without changing its currents and voltages.

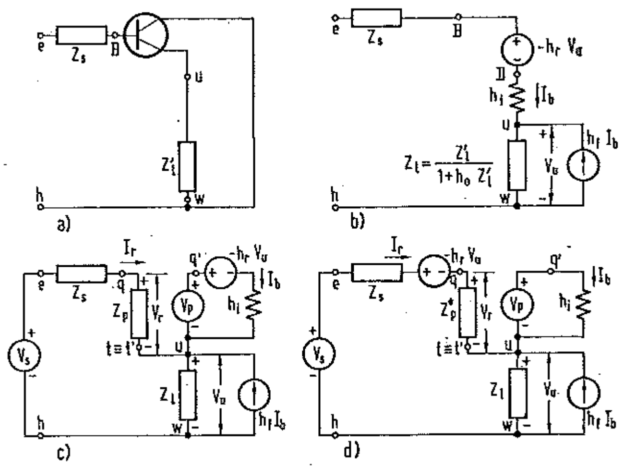

Fig. 5. — a) Common collector ($w \equiv C$) or common emitter ($w \equiv E$) amplifier; b) its a.c. equivalent circuit. Reference node $t \equiv t'$ put in u and control node $q \equiv q'$ made coincide with B and with D in the cases c) and d), respectively.

3. – Examples.

In order to illustrate with an example the preceding theory one considers the network of fig. 5 a whose a.c. equivalent circuit, in terms of hybrid parameters h, is indicated in fig. 5 b. With the input and output ports in $e\,h$ and $u\,w$, respectively, it is a common collecotr (CCA) or common emitter (CEA) amplifier when the node w coincieds with the collector C or the emitter E, respectively, of the transitor.

If the nodes $q \equiv q' \equiv B$ and $t \equiv t' \equiv u$ are chosen as control and reference nodes, respectively, the configuration of fig. 5 c is obtained. With such reference nodes one obtains somewhat elaborate expressions for the various quantities. In fact from (1) and (2) and from fig. 5 c one has:

$$A = h_f\, Z_l\, (Z_s + Z_p)/\Delta \quad , \quad \beta = - Z_p/(Z_s + Z_p) \quad ,$$
$$G_t = (Z_s + Z_l + Z_p)/\Delta \, ,$$
$$\gamma = Z_l\, h_i/\Delta \quad , \quad \alpha = Z_p\, (h_i - h_f\, h_r\, Z_l)/\Delta \quad , \quad \varrho = h_r\, Z_l/\Delta \, ,$$

being:

$$\Delta = h_i\, (Z_s + Z_l + Z_p) - h_f\, h_r\, Z_l\, (Z_s + Z_p) \, .$$

From (9) Z_p becomes:

$$Z_p = \frac{(h_i - h_r\, h_f\, Z_l)\,[h_i + Z_l\,(h_f - h_r - h_r\, h_f)] - h_r\, h_f\, Z_l^2}{h_i + h_f\,(1 - h_r)\, Z_l} \, , \tag{18}$$

which, in the CEA, being [6] $h_{ic} = h_{ie}$; $h_{rc} = 1$, $h_{fc} = -(1 + h_{fe})$ and $h_{oc} = h_{oe}$, gives $Z_{pe} = h_{ie} + h_{fe}\, Z_l$ and in the CCA, with $h_{re} \simeq 0$, $Z_{pc} = h_{ie}$. By substituting the quantities so deduced in the relationships of the preceding analysis one obtains the normal expressions of the gain, of the impedances and so on, which are derived in a more direct way by means of the usual methods. The present analysis serves only to remark that also a common emitter amplifier is a feedback network when the collector is considered as reference node $t \equiv t'$ and the base as control node $q \equiv q'$, that is when the base-collector port is considered as imput port of the « intrinsic » amplifier.

The analysis is remarkably simplified if one chooses as control node $q \equiv q'$ to cut, as indicated in fig. 5 d, the point D of fig. 5 b. In fact in this case $\varrho = 0$ and $Z_i = h_i$, so that, from (9), it is directly $Z_p{}^* = h_i$. In consequence, from (1) and (2), it is $A = h_f\, Z_l\, (h_i + Z_s)/(h_i\, \Delta')$, $\beta = - h_i\, (1 - h_r)/(h_i + Z_s)$, $\gamma = Z_l/\Delta'$ and $\alpha = h_i/\Delta'$ being $\Delta' = Z_s + h_i + (1 - h_r)\, Z_l$. From such expressions of the transfer functions and from (10) the over-all voltage gain becomes $A_f = Z_l\, (1 + h_f)/[h_i + Z_s + Z_l\,(1 - h_r)\,(1 + h_f)]$ from which one obtains the normal results for the CC and CE amplifiers. Finally, being $A\, \beta\, |_{Z^*_p = \infty} = - h_f\, (1 - h_r)\, Z_l/h_i$ and $Z_b = Z_s + Z_l\, (1 - h_r)$, from (16) the input impedance becomes $Z = Z_s + h_i + (1 - h_r)\,(1 + h_f)\, Z_l$. That is all the quantities of the CC and CE amplifiers can be obtained with a single method by means of the generalised elementary feedback theory.

When the backward circuit is distinct from the forward one the control node $q \equiv q'$ to cut and the reference one $t \equiv t'$ are to be chosen along the closed loop in such a way as to simplify the calculations and, in particular, to make $\varrho = 0$.

4. – Conclusions.

It has been shown that a « control » node of a network, arbitraryly chosen, may be cut without changing the voltages and currents if among the new terminals so obtained and a « reference » node, also selected in an arbitrary way, a proper impedance and a proper generator are put.

From the direct analysis, made by means of the normal four-terminal networks and transfer function theories, of such a decomposed open-loop circuit (equivalent to the original one) a general feedback theory, which keeps the intuitive meaning of the elementary one and at the same time may be applied to any type of network, has been obtained.

In particular in this way exact rules have been given to calculate the various quantities such as the loop gain, the sensitivity, the leakage transmittance, the over-hall system gain, the impedances and so on of an amplifier in which the feebdack is purposefully introduced and which is built by means of active elements, such as bipolar junction transistors, characterised by a low imput impedance and by a non negligible intrinsic feedback.

In conclusion the proposed decomposition theorem allows one to use in general the elementary feedback theory without losing its important intuitive meaning.

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